

The LIE Package

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LIE is a package of functions for the classification of real n -dimensional Lie algebras. It consists of two modules: **liendmc1** and **lie1234**.

liendmc1

With the help of the functions in this module real n -dimensional Lie algebras L with a derived algebra $L^{(1)}$ of dimension 1 can be classified. L has to be defined by its structure constants c_{ij}^k in the basis $\{X_1, \dots, X_n\}$ with $[X_i, X_j] = c_{ij}^k X_k$. The user must define an ARRAY LIENSTRUCIN(n, n, n) with n being the dimension of the Lie algebra L . The structure constants LIENSTRUCIN(i, j, k):= c_{ij}^k for $i < j$ should be given. Then the procedure LIENDIMCOM1 can be called. Its syntax is:

LIENDIMCOM1(<number>).

<number> corresponds to the dimension n . The procedure simplifies the structure of L performing real linear transformations. The returned value is a list of the form

- (i) {LIE_ALGEBRA(2), COMMUTATIVE(n-2)} or
- (ii) {HEISENBERG(k), COMMUTATIVE(n-k)}

with $3 \leq k \leq n$, k odd.

The concepts correspond to the following theorem (LIE_ALGEBRA(2) \rightarrow L_2 , HEISENBERG(k) \rightarrow H_k and COMMUTATIVE(n-k) \rightarrow C_{n-k}):

Theorem. Every real n -dimensional Lie algebra L with a 1-dimensional derived algebra can be decomposed into one of the following forms:

- (i) $C(L) \cap L^{(1)} = \{0\}$: $L_2 \oplus C_{n-2}$ or
- (ii) $C(L) \cap L^{(1)} = L^{(1)}$: $H_k \oplus C_{n-k}$ ($k = 2r - 1$, $r \geq 2$), with

1. $C(L) = C_j \oplus (L^{(1)} \cap C(L))$ and $\dim C_j = j$,
2. L_2 is generated by Y_1, Y_2 with $[Y_1, Y_2] = Y_1$,
3. H_k is generated by $\{Y_1, \dots, Y_k\}$ with
 $[Y_2, Y_3] = \dots = [Y_{k-1}, Y_k] = Y_1$.

(cf. [?])

The returned list is also stored as LIE_LIST. The matrix LIENTRANS gives the transformation from the given basis $\{X_1, \dots, X_n\}$ into the standard basis $\{Y_1, \dots, Y_n\}$: $Y_j = (\text{LIENTRANS})_j^k X_k$.

A more detailed output can be obtained by turning on the switch TR_LIE:

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ON TR_LIE;
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before the procedure LIENDIMCOM1 is called.

The returned list could be an input for a data bank in which mathematical relevant properties of the obtained Lie algebras are stored.

lie1234

This part of the package classifies real low-dimensional Lie algebras L of the dimension $n := \dim L = 1, 2, 3, 4$. L is also given by its structure constants c_{ij}^k in the basis $\{X_1, \dots, X_n\}$ with $[X_i, X_j] = c_{ij}^k X_k$. An ARRAY LIESTRIN(n, n, n) has to be defined and LIESTRIN(i, j, k):= c_{ij}^k for $i < j$ should be given. Then the procedure LIECLASS can be performed whose syntax is:

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LIECLASS(<number>).
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<number> should be the dimension of the Lie algebra L . The procedure stepwise simplifies the commutator relations of L using properties of invariance like the dimension of the centre, of the derived algebra, unimodularity etc. The returned value has the form:

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{LIEALG(n), COMTAB(m)},
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where m corresponds to the number of the standard form (basis: $\{Y_1, \dots, Y_n\}$) in an enumeration scheme. The corresponding enumeration schemes are listed below (cf. [?],[?]). In case that the standard form in the enumeration scheme depends on one (or two) parameter(s) p_1 (and p_2) the list is expanded to:

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{LIEALG(n), COMTAB(m), p1, p2}.
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This returned value is also stored as LIE_CLASS. The linear transformation from the basis $\{X_1, \dots, X_n\}$ into the basis of the standard form $\{Y_1, \dots, Y_n\}$ is given by the matrix LIEMAT: $Y_j = (\text{LIEMAT})_j^k X_k$.

By turning on the switch TR_LIE:

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ON TR_LIE;
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before the procedure LIECLASS is called the output contains not only the list LIE_CLASS but also the non-vanishing commutator relations in the standard form.

By the value m and the parameters further examinations of the Lie algebra are possible, especially if in a data bank mathematical relevant properties of the enumerated standard forms are stored.

Enumeration schemes for lie1234

returned list LIE_CLASS	the corresponding commutator relations
LIEALG(1),COMTAB(0)	commutative case
LIEALG(2),COMTAB(0)	commutative case
LIEALG(2),COMTAB(1)	$[Y_1, Y_2] = Y_2$
LIEALG(3),COMTAB(0)	commutative case
LIEALG(3),COMTAB(1)	$[Y_1, Y_2] = Y_3$
LIEALG(3),COMTAB(2)	$[Y_1, Y_3] = Y_3$
LIEALG(3),COMTAB(3)	$[Y_1, Y_3] = Y_1, [Y_2, Y_3] = Y_2$
LIEALG(3),COMTAB(4)	$[Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(5)	$[Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(6)	$[Y_1, Y_3] = -Y_1 + p_1 Y_2, [Y_2, Y_3] = Y_1, p_1 \neq 0$
LIEALG(3),COMTAB(7)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(3),COMTAB(8)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(0)	commutative case
LIEALG(4),COMTAB(1)	$[Y_1, Y_4] = Y_1$
LIEALG(4),COMTAB(2)	$[Y_2, Y_4] = Y_1$
LIEALG(4),COMTAB(3)	$[Y_1, Y_3] = Y_1, [Y_2, Y_4] = Y_2$
LIEALG(4),COMTAB(4)	$[Y_1, Y_3] = -Y_2, [Y_2, Y_4] = Y_2,$ $[Y_1, Y_4] = [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(5)	$[Y_2, Y_4] = Y_2, [Y_1, Y_4] = [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(6)	$[Y_2, Y_4] = Y_1, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(7)	$[Y_2, Y_4] = Y_2, [Y_3, Y_4] = Y_1$
LIEALG(4),COMTAB(8)	$[Y_1, Y_4] = -Y_2, [Y_2, Y_4] = Y_1$
LIEALG(4),COMTAB(9)	$[Y_1, Y_4] = -Y_1 + p_1 Y_2, [Y_2, Y_4] = Y_1, p_1 \neq 0$
LIEALG(4),COMTAB(10)	$[Y_1, Y_4] = Y_1, [Y_2, Y_4] = Y_2$
LIEALG(4),COMTAB(11)	$[Y_1, Y_4] = Y_2, [Y_2, Y_4] = Y_1$

returned list LIE_CLASS	the corresponding commutator relations
LIEALG(4),COMTAB(12)	$[Y_1, Y_4] = Y_1 + Y_2, [Y_2, Y_4] = Y_2 + Y_3,$ $[Y_3, Y_4] = Y_3$
LIEALG(4),COMTAB(13)	$[Y_1, Y_4] = Y_1, [Y_2, Y_4] = p_1 Y_2, [Y_3, Y_4] = p_2 Y_3,$ $p_1, p_2 \neq 0$
LIEALG(4),COMTAB(14)	$[Y_1, Y_4] = p_1 Y_1 + Y_2, [Y_2, Y_4] = -Y_1 + p_1 Y_2,$ $[Y_3, Y_4] = p_2 Y_3, p_2 \neq 0$
LIEALG(4),COMTAB(15)	$[Y_1, Y_4] = p_1 Y_1 + Y_2, [Y_2, Y_4] = p_1 Y_2,$ $[Y_3, Y_4] = Y_3, p_1 \neq 0$
LIEALG(4),COMTAB(16)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = (1 + p_1)Y_2, [Y_3, Y_4] = (1 - p_1)Y_3,$ $p_1 \geq 0$
LIEALG(4),COMTAB(17)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = Y_2 - p_1 Y_3, [Y_3, Y_4] = p_1 Y_2 + Y_3,$ $p_1 \neq 0$
LIEALG(4),COMTAB(18)	$[Y_1, Y_4] = 2Y_1, [Y_2, Y_3] = Y_1,$ $[Y_2, Y_4] = Y_2 + Y_3, [Y_3, Y_4] = Y_3$
LIEALG(4),COMTAB(19)	$[Y_2, Y_3] = Y_1, [Y_2, Y_4] = Y_3, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(20)	$[Y_2, Y_3] = Y_1, [Y_2, Y_4] = -Y_3, [Y_3, Y_4] = Y_2$
LIEALG(4),COMTAB(21)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = -Y_2, [Y_2, Y_3] = Y_1$
LIEALG(4),COMTAB(22)	$[Y_1, Y_2] = Y_3, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1$

References