A Linear Algebra package for REDUCE

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1 Introduction

This package provides a selection of functions that are useful in the world of linear algebra. These functions are described alphabetically in section 3 of this document and are labelled 3.1 to 3.51. They can be classified into four sections(n.b: the numbers after the dots signify the function label in section 3).

Contributions to this package have been made by Walter Tietze (ZIB).

1.1 Basic matrix handling

1.2 Constructors

Functions that create matrices.

1.3 High level algorithms

There is a separate NORMFORM[1] package for computing the following matrix normal forms in REDUCE.

smithex, smithex_int, frobenius, ratjordan, jordansymbolic, jordan.

1.4 Predicates

matrixp	...	3.29	squarep	...	3.42
symmetricp	...	3.49			

Note on examples:

In the examples the matrix $\mathcal{A}$ will be

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Notation

Throughout $\mathcal{I}$ is used to indicate the identity matrix and $\mathcal{A}^T$ to indicate the transpose of the matrix $\mathcal{A}$.

2 Getting started

If you have not used matrices within REDUCE before then the following may be helpful.

Creating matrices

Initialisation of matrices takes the following syntax:

```
mat1 := mat((a,b,c),(d,e,f),(g,h,i));
```

will produce

$$mat1 := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Getting at the entries

The (i,j)'th entry can be accessed by:

```
mat1(i,j);
```

Loading the linear_algebra package

The package is loaded by:

```
load_package linalg;
```

3 What's available

3.1 add_columns, add_rows

`add_columns(`$\mathcal{A}$`,c1,c2,expr);`

$\mathcal{A}$	:-	a matrix.
c1,c2	:-	positive integers.
expr	:-	a scalar expression.

Synopsis:

`add_columns` replaces column c2 of $\mathcal{A}$ by expr $*$ column($\mathcal{A}$,c1) + column($\mathcal{A}$,c2).

`add_rows` performs the equivalent task on the rows of $\mathcal{A}$.

Examples:

$$\texttt{add\_columns}(\mathcal{A},1,2,x) \quad = \quad \begin{pmatrix} 1 & x+2 & 3 \\ 4 & 4*x+5 & 6 \\ 7 & 7*x+8 & 9 \end{pmatrix}$$

$$\texttt{add\_rows}(\mathcal{A},2,3,5) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 27 & 33 & 39 \end{pmatrix}$$

Related functions:

`add_to_columns`, `add_to_rows`, `mult_columns`, `mult_rows`.

3.2 add_rows

see: `add_columns`.

3.3 add_to_columns, add_to_rows

`add_to_columns(`$\mathcal{A}$`,column_list,expr);`

$\mathcal{A}$	:-	a matrix.
column_list	:-	a positive integer or a list of positive integers.
expr	:-	a scalar expression.

Synopsis:

`add_to_columns` adds expr to each column specified in column_list of $\mathcal{A}$.

`add_to_rows` performs the equivalent task on the rows of $\mathcal{A}$.

Examples:

$$\texttt{add\_to\_columns}(\mathcal{A},\{1,2\},10) \quad = \quad \begin{pmatrix} 11 & 12 & 3 \\ 14 & 15 & 6 \\ 17 & 18 & 9 \end{pmatrix}$$

$$\texttt{add\_to\_rows}(\mathcal{A},2,-x) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ -x+4 & -x+5 & -x+6 \\ 7 & 8 & 9 \end{pmatrix}$$

Related functions:

`add_columns`, `add_rows`, `mult_rows`, `mult_columns`.

3.4 add_to_rows

see: `add_to_columns`.

3.5 augment_columns, stack_rows

`augment_columns(`$\mathcal{A}$`,column_list);`

$\mathcal{A}$	:-	a matrix.
column_list	:-	either a positive integer or a list of positive integers.

Synopsis:

`augment_columns` gets hold of the columns of $\mathcal{A}$ specified in column_list and sticks them together.

`stack_rows` performs the same task on rows of $\mathcal{A}$.

Examples:

$$\texttt{augment\_columns}(\mathcal{A},\{1,2\}) \quad = \quad \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\texttt{stack\_rows}(\mathcal{A},\{1,3\}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

Related functions:

`get_columns`, `get_rows`, `sub_matrix`.

3.6 band_matrix

`band_matrix(expr_list,square_size);`

expr_list	:-	either a single scalar expression or a list of an odd number of scalar expressions.
square_size	:-	a positive integer.

Synopsis:

`band_matrix` creates a square matrix of dimension square_size. The diagonal consists of the middle expr of the expr_list. The exprs to the left of this fill the required number of sub_diagonals and the exprs to the right the super_diagonals.

Examples:

$$\texttt{band\_matrix}(\{x,y,z\},6) \quad = \quad \begin{pmatrix} y & z & 0 & 0 & 0 & 0 \\ x & y & z & 0 & 0 & 0 \\ 0 & x & y & z & 0 & 0 \\ 0 & 0 & x & y & z & 0 \\ 0 & 0 & 0 & x & y & z \\ 0 & 0 & 0 & 0 & x & y \end{pmatrix}$$

Related functions:

`diagonal`.

3.7 block_matrix

`block_matrix(r,c,matrix_list);`

r,c	:-	positive integers.
matrix_list	:-	a list of matrices.

Synopsis:

`block_matrix` creates a matrix that consists of r by c matrices filled from the matrix_list row wise.

Examples:

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 22 & 33 \\ 44 & 55 \end{pmatrix}$$

$$\texttt{block\_matrix}(2,3,\{\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{D},\mathcal{C},\mathcal{B}\}) \quad = \quad \begin{pmatrix} 1 & 0 & 5 & 22 & 33 \\ 0 & 1 & 5 & 44 & 55 \\ 22 & 33 & 5 & 1 & 0 \\ 44 & 55 & 5 & 0 & 1 \end{pmatrix}$$

3.8 char_matrix

`char_matrix(`$\mathcal{A},\lambda$`);`

$\mathcal{A}$	:-	a square matrix.
λ	:-	a symbol or algebraic expression.

Synopsis:

`char_matrix` creates the characteristic matrix $\mathcal{C}$ of $\mathcal{A}$.

This is $\mathcal{C} = \lambda * \mathcal{I} - \mathcal{A}$.

Examples:

$$\texttt{char\_matrix}(\mathcal{A}, x) \quad = \quad \begin{pmatrix} x-1 & -2 & -3 \\ -4 & x-5 & -6 \\ -7 & -8 & x-9 \end{pmatrix}$$

Related functions:

`char_poly`.

3.9 char_poly

`char_poly(`$\mathcal{A}, \lambda$`);`

$\mathcal{A}$:- a square matrix.
λ :- a symbol or algebraic expression.

Synopsis:

`char_poly` finds the characteristic polynomial of $\mathcal{A}$.

This is the determinant of $\lambda * \mathcal{I} - \mathcal{A}$.

Examples:

`char_poly(A,`x`)` $= x^3 - 15 * x^2 - 18 * x$

Related functions:

`char_matrix`.

3.10 cholesky

`cholesky(`$\mathcal{A}$`);`

$\mathcal{A}$:- a positive definite matrix containing numeric entries.

Synopsis:

`cholesky` computes the cholesky decomposition of $\mathcal{A}$.

It returns $\{\mathcal{L},\mathcal{U}\}$ where $\mathcal{L}$ is a lower matrix, $\mathcal{U}$ is an upper matrix, $\mathcal{A} = \mathcal{L}\mathcal{U}$, and $\mathcal{U} = \mathcal{L}^T$.

Examples:

$$\mathcal{F} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\texttt{cholesky}(\mathcal{F}) \quad = \quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

Related functions:

`lu_decom`.

3.11 coeff_matrix

`coeff_matrix({lin_eqn`$_1$`,lin_eqn`$_2$`, ...,lin_eqn`$_n$`});` *

lin_eqn$_1$,lin_eqn$_2$, ...,lin_eqn$_n$:- linear equations. Can be of the form *equation* = *number* or just *equation*.

Synopsis:

`coeff_matrix` creates the coefficient matrix $\mathcal{C}$ of the linear equations.

It returns $\{\mathcal{C}, \mathcal{X}, \mathcal{B}\}$ such that $\mathcal{C}\mathcal{X} = \mathcal{B}$.

Examples:

$\texttt{coeff\_matrix}(\{x + y + 4 * z = 10, y + x - z = 20, x + y + 4\}) =$

$$\left\{ \begin{pmatrix} 4 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} z \\ y \\ x \end{pmatrix}, \begin{pmatrix} 10 \\ 20 \\ -4 \end{pmatrix} \right\}$$

3.12 column_dim, row_dim

`column_dim(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix.

Synopsis:

`column_dim` finds the column dimension of $\mathcal{A}$.

*If you're feeling lazy then the {}'s can be omitted.

`row_dim` finds the row dimension of $\mathcal{A}$.

Examples:

`column_dim`$(\mathcal{A}) = 3$

3.13 companion

`companion(poly,x);`

poly	:-	a monic univariate polynomial in x.
x	:-	the variable.

Synopsis:

`companion` creates the companion matrix $\mathcal{C}$ of poly.

This is the square matrix of dimension n, where n is the degree of poly w.r.t. x.

The entries of $\mathcal{C}$ are: $\mathcal{C}$(i,n) = -coeffn(poly,x,i-1) for i = 1 ... n, $\mathcal{C}$(i,i-1) = 1 for i = 2 ... n and the rest are 0.

Examples:

$$\texttt{companion}(x^4 + 17 * x^3 - 9 * x^2 + 11, x) \quad = \quad \begin{pmatrix} 0 & 0 & 0 & -11 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -17 \end{pmatrix}$$

Related functions:

`find_companion`.

3.14 copy_into

`copy_into(`$\mathcal{A},\mathcal{B}$`,r,c);`

$\mathcal{A},\mathcal{B}$	:-	matrices.
r,c	:-	positive integers.

Synopsis:

`copy_into` copies matrix $\mathcal{A}$ into $\mathcal{B}$ with $\mathcal{A}$(1,1) at $\mathcal{B}$(r,c).

Examples:

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\texttt{copy\_into}(\mathcal{A}, \mathcal{G}, 1, 2) \quad = \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ 0 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Related functions:

`augment_columns`, `extend`, `matrix_augment`, `matrix_stack`, `stack_rows`, `sub_matrix`.

3.15 diagonal

`diagonal({mat`$_1$`,mat`$_2$`, ...,mat`$_n$`});`[†]

mat$_1$,mat$_2$, …,mat$_n$:- each can be either a scalar expr or a square matrix.

Synopsis:

`diagonal` creates a matrix that contains the input on the diagonal.

Examples:

$$\mathcal{H} = \begin{pmatrix} 66 & 77 \\ 88 & 99 \end{pmatrix}$$

$$\texttt{diagonal}(\{\mathcal{A}, x, \mathcal{H}\}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 & 0 \\ 7 & 8 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 66 & 77 \\ 0 & 0 & 0 & 0 & 88 & 99 \end{pmatrix}$$

Related functions:

`jordan_block`.

[†]If you're feeling lazy then the {}'s can be omitted.

3.16 extend

`extend(`$\mathcal{A}$`,r,c,expr);`

$\mathcal{A}$	:-	a matrix.
r,c	:-	positive integers.
expr	:-	algebraic expression or symbol.

Synopsis:

`extend` returns a copy of $\mathcal{A}$ that has been extended by r rows and c columns. The new entries are made equal to expr.

Examples:

$$\texttt{extend}(\mathcal{A},1,2,x) \quad = \quad \begin{pmatrix} 1 & 2 & 3 & x & x \\ 4 & 5 & 6 & x & x \\ 7 & 8 & 9 & x & x \\ x & x & x & x & x \end{pmatrix}$$

Related functions:

`copy_into`, `matrix_augment`, `matrix_stack`, `remove_columns`, `remove_rows`.

3.17 find_companion

`find_companion(`$\mathcal{A}$`,x);`

$\mathcal{A}$	:-	a matrix.
x	:-	the variable.

Synopsis:

Given a companion matrix, `find_companion` finds the polynomial from which it was made.

Examples:

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & -11 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -17 \end{pmatrix}$$

$$\texttt{find\_companion}(\mathcal{C},x) = x^4 + 17 * x^3 - 9 * x^2 + 11$$

Related functions:

`companion`.

3.18 get_columns, get_rows

`get_columns(`$\mathcal{A}$`,column_list);`

$\mathcal{A}$	:-	a matrix.
c	:-	either a positive integer or a list of positive integers.

Synopsis:

`get_columns` removes the columns of $\mathcal{A}$ specified in column_list and returns them as a list of column matrices.

`get_rows` performs the same task on the rows of $\mathcal{A}$.

Examples:

$$\texttt{get\_columns}(\mathcal{A},\{1,3\}) \quad = \quad \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}$$

$$\texttt{get\_rows}(\mathcal{A},2) \quad = \quad \left\{ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \right\}$$

Related functions:

`augment_columns`, `stack_rows`, `sub_matrix`.

3.19 get_rows

see: `get_columns`.

3.20 gram_schmidt

`gram_schmidt(`$\{$`vec`$_1$`,vec`$_2$`, ...,vec`$_n\}$`);` [‡]

[‡]If you're feeling lazy then the {}'s can be omitted.

$\text{vec}_1, \text{vec}_2, \ldots, \text{vec}_n$:- linearly independent vectors. Each vector must be written as a list, eg:{1,0,0}.

Synopsis:

`gram_schmidt` performs the gram_schmidt orthonormalisation on the input vectors.

It returns a list of orthogonal normalised vectors.

Examples:

`gram_schmidt({{1,0,0},{1,1,0},{1,1,1}})` $= \{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

`gram_schmidt({{1,2},{3,4}})` $= \{\{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\}, \{\frac{2*\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\}\}$

3.21 hermitian_tp

`hermitian_tp(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix.

Synopsis:

`hermitian_tp` computes the hermitian transpose of $\mathcal{A}$.

This is a matrix in which the (i, j)'th entry is the conjugate of the (j, i)'th entry of $\mathcal{A}$.

Examples:

$$\mathcal{J} = \begin{pmatrix} i+1 & i+2 & i+3 \\ 4 & 5 & 2 \\ 1 & i & 0 \end{pmatrix}$$

`hermitian_tp(`$\mathcal{J}$`)` $= \begin{pmatrix} -i+1 & 4 & 1 \\ -i+2 & 5 & -i \\ -i+3 & 2 & 0 \end{pmatrix}$

Related functions:

`tp`[§].

[§]standard reduce call for the transpose of a matrix - see REDUCE User's Manual[2].

3.22 hessian

`hessian(expr,variable_list);`

expr	:-	a scalar expression.
variable_list	:-	either a single variable or a list of variables.

Synopsis:

`hessian` computes the hessian matrix of expr w.r.t. the varibles in variable_list.

This is an n by n matrix where n is the number of variables and the (i, j)'th entry is df(expr,variable_list(i),variable_list(j)).

Examples:

$$\texttt{hessian}(x*y*z+x^2,\{w,x,y,z\}) \quad = \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & z & y \\ 0 & z & 0 & x \\ 0 & y & x & 0 \end{pmatrix}$$

Related functions:

`df`[¶].

3.23 hilbert

`hilbert(square_size,expr);`

square_size	:-	a positive integer.
expr	:-	an algebraic expression.

Synopsis:

`hilbert` computes the square hilbert matrix of dimension square_size.

This is the symmetric matrix in which the (i, j)'th entry is 1/(i+j-expr).

Examples:

$$\texttt{hilbert}(3,y+x) \quad = \quad \begin{pmatrix} \frac{-1}{x+y-2} & \frac{-1}{x+y-3} & \frac{-1}{x+y-4} \\ \frac{-1}{x+y-3} & \frac{-1}{x+y-4} & \frac{-1}{x+y-5} \\ \frac{-1}{x+y-4} & \frac{-1}{x+y-5} & \frac{-1}{x+y-6} \end{pmatrix}$$

[¶]standard reduce call for differentiation - see REDUCE User's Manual[2]

3.24 jacobian

`jacobian(expr_list,variable_list);`

expr_list	:-	either a single algebraic expression or a list of algebraic expressions.
variable_list	:-	either a single variable or a list of variables.

Synopsis:

`jacobian` computes the jacobian matrix of expr_list w.r.t. variable_list.

This is a matrix whose (i, j)'th entry is df(expr_list(i), variable_list(j)).

The matrix is n by m where n is the number of variables and m the number of expressions.

Examples:

`jacobian(`$\{x^4, x*y^2, x*y*z^3\}$`,`$\{w,x,y,z\}$`)` $=$

$$\begin{pmatrix} 0 & 4*x^3 & 0 & 0 \\ 0 & y^2 & 2*x*y & 0 \\ 0 & y*z^3 & x*z^3 & 3*x*y*z^2 \end{pmatrix}$$

Related functions:

`hessian`, `df`[‖].

3.25 jordan_block

`jordan_block(expr,square_size);`

expr	:-	an algebraic expression or symbol.
square_size	:-	a positive integer.

Synopsis:

`jordan_block` computes the square jordan block matrix $\mathcal{J}$ of dimension square_size.

The entries of $\mathcal{J}$ are: $\mathcal{J}$(i,i) = expr for i=1 … n, $\mathcal{J}$(i,i+1) = 1 for i=1 … n-1, and all other entries are 0.

[‖] standard reduce call for differentiation - see REDUCE User's Manual[2].

Examples:

$$\texttt{jordan\_block}(\texttt{x},5) \quad = \quad \begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & x \end{pmatrix}$$

Related functions:

`diagonal`, `companion`.

3.26 lu_decom

`lu_decom(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix containing either numeric entries or imaginary entries with numeric coefficients.

Synopsis:

`lu_decom` performs LU decomposition on $\mathcal{A}$, ie: it returns $\{\mathcal{L},\mathcal{U}\}$ where $\mathcal{L}$ is a lower diagonal matrix, $\mathcal{U}$ an upper diagonal matrix and $\mathcal{A} = \mathcal{L}\mathcal{U}$.

caution:

The algorithm used can swap the rows of $\mathcal{A}$ during the calculation. This means that $\mathcal{L}\mathcal{U}$ does not equal $\mathcal{A}$ but a row equivalent of it. Due to this, `lu_decom` returns $\{\mathcal{L},\mathcal{U},\text{vec}\}$. The call `convert(`$\mathcal{A}$`,vec)` will return the matrix that has been decomposed, ie: $\mathcal{L}\mathcal{U} =$ `convert(`$\mathcal{A}$`,vec)`.

Examples:

$$\mathcal{K} = \begin{pmatrix} 1 & 3 & 5 \\ -4 & 3 & 7 \\ 8 & 6 & 4 \end{pmatrix}$$

$$\texttt{lu} := \texttt{lu\_decom}(\mathcal{K}) \quad = \quad \left\{ \begin{pmatrix} 8 & 0 & 0 \\ -4 & 6 & 0 \\ 1 & 2.25 & 1.1251 \end{pmatrix}, \begin{pmatrix} 1 & 0.75 & 0.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}, [\,3\;2\;3\,] \right\}$$

$$\texttt{first lu * second lu} \quad = \quad \begin{pmatrix} 8 & 6 & 4 \\ -4 & 3 & 7 \\ 1 & 3 & 5 \end{pmatrix}$$

$$\texttt{convert(}\mathcal{K}\texttt{,third lu)} \quad = \quad \begin{pmatrix} 8 & 6 & 4 \\ -4 & 3 & 7 \\ 1 & 3 & 5 \end{pmatrix}$$

$$\mathcal{P} = \begin{pmatrix} i+1 & i+2 & i+3 \\ 4 & 5 & 2 \\ 1 & i & 0 \end{pmatrix}$$

$$\texttt{lu} := \texttt{lu\_decom}(\mathcal{P}) \quad = \quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 4 & -4*i+5 & 0 \\ i+1 & 3 & 0.41463*i+2.26829 \end{pmatrix}, \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0.19512*i+0.24390 \\ 0 & 0 & 1 \end{pmatrix}, [\,3\ 2\ 3\,] \right\}$$

$$\texttt{first lu * second lu} \quad = \quad \begin{pmatrix} 1 & i & 0 \\ 4 & 5 & 2 \\ i+1 & i+2 & i+3 \end{pmatrix}$$

$$\texttt{convert(P,third lu)} \quad = \quad \begin{pmatrix} 1 & i & 0 \\ 4 & 5 & 2 \\ i+1 & i+2 & i+3 \end{pmatrix}$$

Related functions:

`cholesky`.

3.27 make_identity

`make_identity(square_size);`

square_size :- a positive integer.

Synopsis:

`make_identity` creates the identity matrix of dimension square_size.

Examples:

$$\texttt{make\_identity}(4) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Related functions:

`diagonal`.

3.28 matrix_augment, matrix_stack

`matrix_augment({mat`$_1$`,mat`$_2$`, ...,mat`$_n$`});`[**]

mat$_1$,mat$_2$, ...,mat$_n$:- matrices.

Synopsis:

`matrix_augment` sticks the matrices in matrix_list together horizontally.

`matrix_stack` sticks the matrices in matrix_list together vertically.

Examples:

$$\texttt{matrix\_augment}(\{\mathcal{A},\mathcal{A}\}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 4 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{pmatrix}$$

$$\texttt{matrix\_stack}(\{\mathcal{A},\mathcal{A}\}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Related functions:

`augment_columns`, `stack_rows`, `sub_matrix`.

3.29 matrixp

`matrixp(test_input);`

[**]If you're feeling lazy then the {}'s can be omitted.

test_input	:-	anything you like.

Synopsis:

`matrixp` is a boolean function that returns t if the input is a matrix and nil otherwise.

Examples:

`matrixp`($\mathcal{A}$) = t

`matrixp`(doodlesackbanana) = nil

Related functions:

`squarep`, `symmetricp`.

3.30 matrix_stack

see: `matrix_augment`.

3.31 minor

`minor(`$\mathcal{A}$`,r,c);`

$\mathcal{A}$	:-	a matrix.
r,c	:-	positive integers.

Synopsis:

`minor` computes the (r,c)'th minor of $\mathcal{A}$.

This is created by removing the r'th row and the c'th column from $\mathcal{A}$.

Examples:

$$\texttt{minor}(\mathcal{A},1,3) \quad = \quad \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

Related functions:

`remove_columns`, `remove_rows`.

3.32 mult_columns, mult_rows

`mult_columns(`$\mathcal{A}$`,column_list,expr);`

$\mathcal{A}$	:-	a matrix.
column_list	:-	a positive integer or a list of positive integers.
expr	:-	an algebraic expression.

Synopsis:

`mult_columns` returns a copy of $\mathcal{A}$ in which the columns specified in column_list have been multiplied by expr.

`mult_rows` performs the same task on the rows of $\mathcal{A}$.

Examples:

$$\texttt{mult\_columns}(\mathcal{A},\{1,3\},x) \quad = \quad \begin{pmatrix} x & 2 & 3*x \\ 4*x & 5 & 6*x \\ 7*x & 8 & 9*x \end{pmatrix}$$

$$\texttt{mult\_rows}(\mathcal{A},2,10) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 40 & 50 & 60 \\ 7 & 8 & 9 \end{pmatrix}$$

Related functions:

`add_to_columns`, `add_to_rows`.

3.33 `mult_rows`

see: `mult_columns`.

3.34 pivot

`pivot(`$\mathcal{A}$`,r,c);`

$\mathcal{A}$	:-	a matrix.
r,c	:-	positive integers such that $\mathcal{A}$(r,c) neq 0.

Synopsis:

`pivot` pivots $\mathcal{A}$ about its (r,c)'th entry.

To do this, multiples of the r'th row are added to every other row in the matrix.

This means that the c'th column will be 0 except for the (r,c)'th entry.

Examples:

$$\texttt{pivot}(\mathcal{A},2,3) \quad = \quad \begin{pmatrix} -1 & -0.5 & 0 \\ 4 & 5 & 6 \\ 1 & 0.5 & 0 \end{pmatrix}$$

Related functions:

`rows_pivot`.

3.35 pseudo_inverse

`pseudo_inverse(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix.

Synopsis:

`pseudo_inverse`, also known as the Moore-Penrose inverse, computes the pseudo inverse of $\mathcal{A}$.

Given the singular value decomposition of $\mathcal{A}$, i.e: $\mathcal{A} = \mathcal{U} \sum \mathcal{V}^T$, then the pseudo inverse $\mathcal{A}^{-1}$ is defined by $\mathcal{A}^{-1} = \mathcal{V}^T \sum^{-1} \mathcal{U}$.

Thus $\mathcal{A} * \texttt{pseudo\_inverse}(\mathcal{A}) = \mathcal{I}$.

Examples:

$$\texttt{pseudo\_inverse}(\mathcal{A}) \quad = \quad \begin{pmatrix} -0.2 & 0.1 \\ -0.05 & 0.05 \\ 0.1 & 0 \\ 0.25 & -0.05 \end{pmatrix}$$

Related functions:

`svd`.

3.36 random_matrix

`random_matrix(r,c,limit);`

r,c, limit :- positive integers.

Synopsis:

`random_matrix` creates an r by c matrix with random entries in the range

$-$limit < entry < limit.

switches:

`imaginary`	:-	if on then matrix entries are x+i*y where $-$limit < x,y < limit.
`not_negative`	:-	if on then 0 < entry < limit. In the imaginary case we have 0 < x,y < limit.
`only_integer`	:-	if on then each entry is an integer. In the imaginary case x and y are integers.
`symmetric`	:-	if on then the matrix is symmetric.
`upper_matrix`	:-	if on then the matrix is upper triangular.
`lower_matrix`	:-	if on then the matrix is lower triangular.

Examples:

$$\texttt{random\_matrix}(3,3,10) \quad = \quad \begin{pmatrix} -4.729721 & 6.987047 & 7.521383 \\ -5.224177 & 5.797709 & -4.321952 \\ -9.418455 & -9.94318 & -0.730980 \end{pmatrix}$$

`on only_integer, not_negative, upper_matrix, imaginary;`

$$\texttt{random\_matrix}(4,4,10) \quad = \quad \begin{pmatrix} 2*i+5 & 3*i+7 & 7*i+3 & 6 \\ 0 & 2*i+5 & 5*i+1 & 2*i+1 \\ 0 & 0 & 8 & i \\ 0 & 0 & 0 & 5*i+9 \end{pmatrix}$$

3.37 remove_columns, remove_rows

`remove_columns(`$\mathcal{A}$`,column_list);`

$\mathcal{A}$	:-	a matrix.
column_list	:-	either a positive integer or a list of positive integers.

Synopsis:

`remove_columns` removes the columns specified in column_list from $\mathcal{A}$.

`remove_rows` performs the same task on the rows of $\mathcal{A}$.

Examples:

$$\texttt{remove\_columns}(\mathcal{A},2) \quad = \quad \begin{pmatrix} 1 & 3 \\ 4 & 6 \\ 7 & 9 \end{pmatrix}$$

$$\texttt{remove\_rows}(\mathcal{A},\{1,3\}) \quad = \quad \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

Related functions:

`minor`.

3.38 remove_rows

see: `remove_columns`.

3.39 row_dim

see: `column_dim`.

3.40 rows_pivot

`rows_pivot(`$\mathcal{A}$`,r,c,{row_list});`

$\mathcal{A}$	:-	a matrix.
r,c	:-	positive integers such that $\mathcal{A}$(r,c) neq 0.
row_list	:-	positive integer or a list of positive integers.

Synopsis:

`rows_pivot` performs the same task as `pivot` but applies the pivot only to the rows specified in row_list.

Examples:

$$\mathcal{N} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\texttt{rows\_pivot}(\mathcal{N}, 2, 3, \{4, 5\}) \quad = \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ -0.75 & 0 & 0.75 \\ -0.375 & 0 & 0.375 \end{pmatrix}$$

Related functions:

`pivot`.

3.41 simplex

`simplex(max/min,objective function,{linear inequalities});`

max/min	:-	either max or min (signifying maximise and minimise).
objective function	:-	the function you are maximising or minimising.
linear inequalities	:-	the constraint inequalities. Each one must be of the form *sum of variables (<=, =, >=) number*.

Synopsis:

`simplex` applies the revised simplex algorithm to find the optimal(either maximum or minimum) value of the objective function under the linear inequality constraints.

It returns {optimal value,{ values of variables at this optimal}}.

The algorithm implies that all the variables are non-negative.

Examples:

`simplex(`$max, x + y, \{x >= 10, y >= 20, x + y <= 25\}$`);`

`***** Error in simplex:  Problem has no feasible solution.`

`simplex(`$max, 10x + 5y + 5.5z, \{5x + 3z <= 200, x + 0.1y + 0.5z <= 12,$
$0.1x + 0.2y + 0.3z <= 9, 30x + 10y + 50z <= 1500\}$`);`

$\{525.0, \{x = 40.0, y = 25.0, z = 0\}\}$

3.42 squarep

`squarep(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix.

Synopsis:

`squarep` is a boolean function that returns t if the matrix is square and nil otherwise.

Examples:

$\mathcal{L} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$

`squarep`$(\mathcal{A})$ = t

`squarep`$(\mathcal{L})$ = nil

Related functions:

`matrixp`, `symmetricp`.

3.43 stack_rows

see: `augment_columns`.

3.44 sub_matrix

`sub_matrix(`$\mathcal{A}$`,row_list,column_list);`

$\mathcal{A}$	:-	a matrix.
row_list, column_list	:-	either a positive integer or a list of positive integers.

Synopsis:

`sub_matrix` produces the matrix consisting of the intersection of the rows specified in row_list and the columns specified in column_list.

Examples:

`sub_matrix`$(\mathcal{A}, \{1,3\}, \{2,3\}) \quad = \quad \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix}$

Related functions:

`augment_columns`, `stack_rows`.

3.45 svd (singular value decomposition)

`svd(`$\mathcal{A}$`);`

$\mathcal{A}$:- a matrix containing only numeric entries.

Synopsis:

`svd` computes the singular value decomposition of $\mathcal{A}$.

It returns $\{\mathcal{U}, \sum, \mathcal{V}\}$ where $\mathcal{A} = \mathcal{U} \sum \mathcal{V}^T$ and $\sum = diag(\sigma_1, \ldots, \sigma_n)$. σ_i for $i = (1 \ldots n)$ are the singular values of $\mathcal{A}$.

n is the column dimension of $\mathcal{A}$.

The singular values of $\mathcal{A}$ are the non-negative square roots of the eigenvalues of $\mathcal{A}^T\mathcal{A}$.

$\mathcal{U}$ and $\mathcal{V}$ are such that $\mathcal{U}\mathcal{U}^T = \mathcal{V}\mathcal{V}^T = \mathcal{V}^T\mathcal{V} = \mathcal{I}_n$.

Examples:

$$\mathcal{Q} = \begin{pmatrix} 1 & 3 \\ -4 & 3 \end{pmatrix}$$

$$\texttt{svd}(\mathcal{Q}) = \left\{ \begin{pmatrix} 0.289784 & 0.957092 \\ -0.957092 & 0.289784 \end{pmatrix}, \begin{pmatrix} 5.149162 & 0 \\ 0 & 2.913094 \end{pmatrix}, \begin{pmatrix} -0.687215 & 0.726453 \\ -0.726453 & -0.687215 \end{pmatrix} \right\}$$

3.46 swap_columns, swap_rows

`swap_columns(`$\mathcal{A}$`,c1,c2);`

$\mathcal{A}$:- a matrix.
c1,c1 :- positive integers.

Synopsis:

`swap_columns` swaps column c1 of $\mathcal{A}$ with column c2.

`swap_rows` performs the same task on 2 rows of $\mathcal{A}$.

Examples:

$$\texttt{swap\_columns}(\mathcal{A},2,3) \quad = \quad \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{pmatrix}$$

Related functions:

`swap_entries`.

3.47 swap_entries

`swap_entries(`$\mathcal{A}$`,{r1,c1},{r2,c2});`

$\mathcal{A}$	:-	a matrix.
r1,c1,r2,c2	:-	positive integers.

Synopsis:

`swap_entries` swaps $\mathcal{A}$(r1,c1) with $\mathcal{A}$(r2,c2).

Examples:

$$\texttt{swap\_entries}(\mathcal{A},\{1,1\},\{3,3\}) \quad = \quad \begin{pmatrix} 9 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}$$

Related functions:

`swap_columns`, `swap_rows`.

3.48 swap_rows

see: `swap_columns`.

3.49 symmetricp

`symmetricp(`$\mathcal{A}$`);`

$\mathcal{A}$	:-	a matrix.

Synopsis:

`symmetricp` is a boolean function that returns t if the matrix is symmetric and nil otherwise.

Examples:

$$\mathcal{M} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

`symmetricp`($\mathcal{A}$) = nil

`symmetricp`($\mathcal{M}$) = t

Related functions:

`matrixp`, `squarep`.

3.50 toeplitz

`toeplitz({expr`$_1$`,expr`$_2$`, ...,expr`$_n$`});` [††]

expr$_1$,expr$_2$, ...,expr$_n$:- algebraic expressions.

Synopsis:

`toeplitz` creates the toeplitz matrix from the expression list.

This is a square symmetric matrix in which the first expression is placed on the diagonal and the i'th expression is placed on the (i-1)'th sub and super diagonals.

It has dimension n where n is the number of expressions.

Examples:

$$\texttt{toeplitz}(\{w,x,y,z\}) = \begin{pmatrix} w & x & y & z \\ x & w & x & y \\ y & x & w & x \\ z & y & x & w \end{pmatrix}$$

3.51 triang_adjoint

`triang_adjoint`($\mathcal{A}$)`;`

[††]If you're feeling lazy then the {}'s can be omitted.

$\mathcal{A}$:- a matrix.

Synopsis:

triang_adjoint computes the triangularizing adjoint $\mathcal{F}$ of matrix $\mathcal{A}$ due to the algorithm of Arne Storjohann. $\mathcal{F}$ is lower triangular matrix and the resulting matrix $\mathcal{T}$ of $\mathcal{F} * \mathcal{A} = \mathcal{T}$ is upper triangular with the property that the i-th entry in the diagonal of $\mathcal{T}$ is the determinant of the principal i-th submatrix of the matrix $\mathcal{A}$.

Examples:

$$\texttt{triang\_adjoint}(\mathcal{A}) = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 6 & -3 \end{pmatrix}$$

$$\mathcal{F} * \mathcal{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

3.52 Vandermonde

vandermonde({expr_1,expr_2, ...,$\text{expr}_\mathtt{n}$}); ††

expr_1,expr_2, ...,$\text{expr}_\mathtt{n}$:- algebraic expressions.

Synopsis:

Vandermonde creates the Vandermonde matrix from the expression list.

This is the square matrix in which the (i, j)'th entry is expr_list(i) $^{(j-1)}$.

It has dimension n where n is the number of expressions.

Examples:

$$\texttt{vandermonde}(\{x, 2*y, 3*z\}) = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2*y & 4*y^2 \\ 1 & 3*z & 9*z^2 \end{pmatrix}$$

3.53 kronecker_product

kronecker_product(Mat_1, Mat_2)

Mat_1, Mat_2 :- Matrices

Synopsis:

`kronecker_product` creates a matrix containing the Kronecker product (also called `direct product` or `tensor product`) of its arguments.

Examples:

```
a1 := mat((1,2),(3,4),(5,6))$
a2 := mat((1,1,1),(2,z,2),(3,3,3))$
kronecker_product(a1,a2);
```

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & z & 2 & 4 & 2*z & 4 \\ 3 & 3 & 3 & 6 & 6 & 6 \\ 3 & 3 & 3 & 4 & 4 & 4 \\ 6 & 3*z & 6 & 8 & 4*z & 8 \\ 9 & 9 & 9 & 12 & 12 & 12 \\ 5 & 5 & 5 & 6 & 6 & 6 \\ 10 & 5*z & 10 & 12 & 6*z & 12 \\ 15 & 15 & 15 & 18 & 18 & 18 \end{pmatrix}$$

4 Fast Linear Algebra

By turning the `fast_la` switch on, the speed of the following functions will be increased:

add_columns	add_rows	augment_columns	column_dim
copy_into	make_identity	matrix_augment	matrix_stack
minor	mult_column	mult_row	pivot
remove_columns	remove_rows	rows_pivot	squarep
stack_rows	sub_matrix	swap_columns	swap_entries
swap_rows	symmetricp		

The increase in speed will be insignificant unless you are making a significant number(i.e: thousands) of calls. When using this switch, error checking is minimised. This means that illegal input may give strange error messages. Beware.

5 Acknowledgments

Many of the ideas for this package came from the Maple[3] Linalg package [4].

The algorithms for `cholesky`, `lu_decom`, and `svd` are taken from the book Linear Algebra - J.H. Wilkinson & C. Reinsch[5].

The `gram_schmidt` code comes from Karin Gatermann's Symmetry package[6] for REDUCE.

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